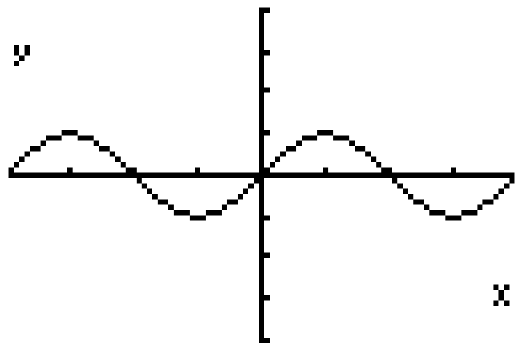
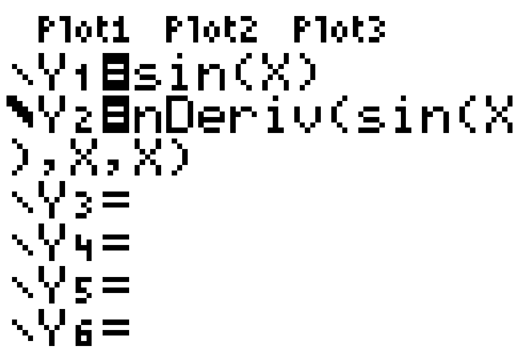
Derivatives of Trigonometric Functions

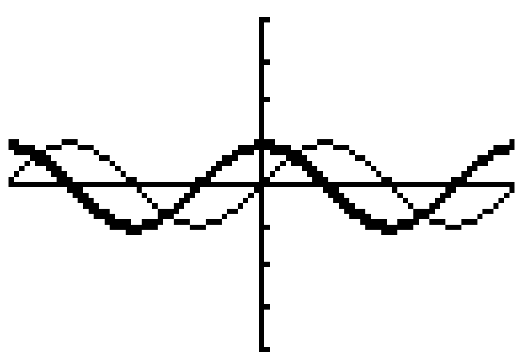
-Graph the function $y = \sin(x)$



-The graphing calculator has a function called **nDeriv** that will graph the numerical derivative of a function at every value of x.

-Using this, plot the numerical derivative of $\sin(x)$.





-What does this look like? '$\cos(x)$

-To verify this we need to go back to the angle sum identity

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

-Now confirm

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{(\sin(x))(\cos(h)-1)+\cos(x)\sin(h)}{h}$$

$$= \lim_{h\to 0}\sin(x) \bullet \lim_{h\to 0}\frac{(\cos(h)-1)}{h} + \lim_{h\to 0}\cos(x) \bullet \lim_{h\to 0}\frac{\sin(h)}{h}$$

NOTE: $\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$

$$= \sin(x) \bullet 0 + \cos(x) \bullet 1$$

$$= \cos(x)$$

-In short, the derivative of sine is cosine.

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Derivative of the Cosine Function

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Example

Find the derivative of $y = x^2 \sin(x)$

$$\frac{dy}{dx} = x^2 \bullet \frac{d}{dx}(\sin(x)) + \sin(x) \bullet \frac{d}{dx}(x^2)$$

$$= x^2 \cos(x) + 2x \sin(x)$$

Example

Find $\frac{dy}{dx}$ if $y = \tan(x)$

$$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Derivatives of Trig Functions

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Finding Tangent and Normal Lines

Find the equation tangent and normal to $y = \frac{\tan x}{x}$ at the point $x = 2$.

$$f'(x) = \frac{x\left(\sec^2 x\right) - \left(\tan x\right)\left(1\right)}{x^2}$$

$$f'(2) = \frac{2\left(\sec^2 2\right) - \left(\tan 2\right)\left(1\right)}{2^2} = 3.43$$

-Tangent

$$y - \left(-1.0925\right) = 3.43\left(x - 2\right)$$

$$y + 1.0925 = 3.43x - 6.86$$

$$y = 3.43x - 7.9525$$

-Normal

$$y - \left(-1.0925\right) = -0.2915\left(x - 2\right)$$

$$y = -0.2915x - 0.5095$$

Example

Find the derivative of $u = \dfrac{\cos x}{1 - \sin x}$

$$\frac{du}{dx} = \frac{(1 - \sin x) \bullet \frac{d}{dx}(\cos x) - \cos x \bullet \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2}$$

$$= \frac{-\sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

Simple Harmonic Motion

-A weight on a spring is stretched 5 units beyond rest $(s = 0)$ and released at $t = 0$.

-Its position is given by $s = 5\cos t$

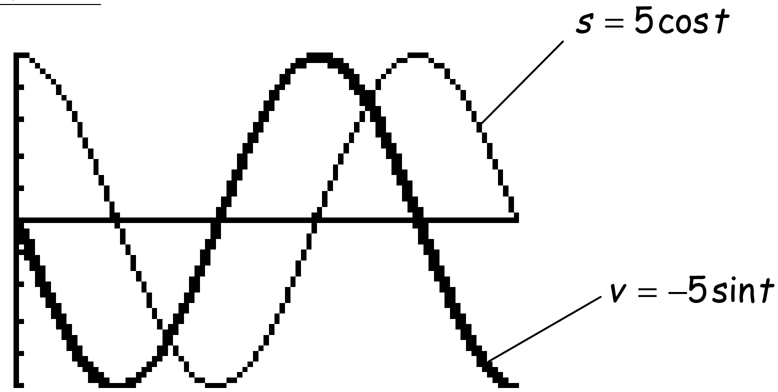
-What is the velocity and acceleration at time t?

Position: $s = 5\cos t$

Velocity: $v = \dfrac{ds}{dt} = \dfrac{d}{dx}(5\cos t) = -5\sin t$

Acceleration: $a = \frac{dv}{dt} = \frac{d}{dx}\left(-5\sin t\right) = -5\cos t$

Information



1) As time passes the weight moves down and up between $s = -5$ and $s = 5$ on the s-axis.

-The amplitude of the motion is 5.
-The period of the motion is 2π .

2) The velocity $v = -5\sin t$ attains the greatest magnitude 5, when $\cos t = 0$. Hence speed $\left|v\right| = 5\left|\sin t\right|$, is greatest when $\cos t = 0$ that is when $s = 0$.

-The speed is zero when $\sin t = 0$.
-This occurs when $s = 5\cos t = \pm 5$

3) The acceleration is always the exact opposite of the position value.

4) Acceleration $a = -5\cos t$ is zero only at the rest position.

-Gravity forces equal to the force of the spring.

Jerk

-A sudden acceleration is called a "jerk"
-A jerk spills a drink in a car.

-The derivative responsible for jerk is the 3rd derivative of position.

-Jerk is the derivative of acceleration.

$$j(t) = \frac{da}{dt} = \frac{d^3 s}{dt^3}$$

-A jerk is thought to cause motion sickness.

A Couple of Jerks

-The jerk caused by the constant acceleration of gravity $g = -32 \text{ ft}/\text{sec}^2$ is 0.

$$j = \frac{d}{dt}(g) = 0$$

-This is why you don't get sick at 1g.
-The jerk of the motion of the spring mass is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t)$$

$$= 5\sin t$$

-Greatest magnitude is when $\sin t = \pm 1$
-This occurs at the rest position.
-When the acceleration changes direction and sign.